

Orbital Functional Geometry XVIII

Orbital Geodesics on Riemannian Manifolds:
Curved Eikonal, Jacobi Fields, and Conformal Orbital Metric

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Abstract

We develop the theory of orbital geodesics on a Riemannian manifold (M, g) , extending the flat eikonal structure of Paper XVII to curved space.

The orbital eikonal equation on (M, g) is:

$$|\nabla_g \phi|_g^2 = \frac{Q(x)^2}{a^2}$$

Its characteristics are the *orbital geodesics*: curves γ satisfying the covariant equation $D\dot{\gamma}/dt = (Q/a^2)\nabla_g Q$. In the absence of Q , these reduce to the geodesics of (M, g) ; the orbital invariant Q introduces a force $\nabla_g Q^2/2a^2$ that deviates trajectories from geodesic motion.

The Jacobi equation for orbital geodesic deviation combines the Riemannian curvature with the Hessian of Q : stability is governed by the competition between K_g and $\text{Hess}_g(Q)$.

On S^2 with $f = e^{\cos \theta}$: the north pole is an orbital attractor, the equator is a free orbital geodesic ($Q = 0$, no deviation), and the south pole is an orbital repeller.

The orbital geodesics are the geodesics of the *conformal orbital metric* $\tilde{g} = (Q^2/a^2)g$, and satisfy the orbital Fermat principle: they extremise $\int_\gamma |Q|/a \, ds_g$. On surfaces, the curvature of $\tilde{g}$ is:

$$\tilde{K} = \frac{a^2}{Q^2}(K_g - \Delta_g \ln |Q|)$$

The zeros of Q are conformal collapse points: they are at zero $\tilde{g}$ -distance from every neighbouring point.

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Introduction

Paper XVII introduced orbital geodesics in $\mathbb{R}^n$ as the characteristics of the orbital eikonal equation $|\nabla\phi|^2 = Q^2/a^2$. The natural next step is to place this structure on a curved Riemannian manifold (M, g) .

On (M, g) , the flat gradient ∇ is replaced by the Riemannian gradient ∇_g , and the ordinary derivative $\dot{\gamma}$ is replaced by the covariant derivative $D\dot{\gamma}/dt$. The Christoffel symbols of (M, g) enter the geodesic equation and couple the orbital force $\nabla_g Q$ to the background curvature.

The main new structure is the conformal orbital metric $\tilde{g} = (Q^2/a^2)g$: orbital geodesics are exactly the geodesics of $\tilde{g}$, and the orbital Fermat principle identifies them as paths that extremise the Q -weighted arc length. The curvature of $\tilde{g}$ mixes the Riemannian curvature K_g with the Laplacian of $\ln|Q|$, showing how the orbital invariant modifies the geometry of (M, g) .

1 Orbital Eikonal on (M, g)

Definition 1 (Orbital eikonal on (M, g)). *Let $\phi : (M, g) \rightarrow \mathbb{R}$ be smooth. The orbital eikonal equation on (M, g) is:*

$$|\nabla_g \phi(x)|_g^2 = g^{ij}(x) \partial_i \phi(x) \partial_j \phi(x) = \frac{Q(x)^2}{a^2}$$

where $Q = \ln f/P - b$ is the orbital invariant.

Theorem 1 (Orbital geodesics on (M, g)). *The characteristics of the orbital eikonal equation are curves $\gamma : I \rightarrow M$ satisfying:*

$$\frac{D\dot{\gamma}}{dt} = \frac{Q(\gamma)}{a^2} \nabla_g Q(\gamma) = \frac{1}{2a^2} \nabla_g Q^2(\gamma)$$

where D/dt is the covariant derivative along γ in (M, g) . These are the orbital geodesics on (M, g) .

Proof. Along a characteristic γ with $\dot{\gamma} = \nabla_g \phi$, covariant differentiation gives $D\dot{\gamma}/dt = \nabla_g(\frac{1}{2}|\nabla_g \phi|_g^2) = \nabla_g(Q^2/2a^2) = (Q/a^2)\nabla_g Q$. $\square$

Remark 1 (Decomposition). *Write the orbital geodesic equation as:*

$$\frac{D\dot{\gamma}}{dt} = \underbrace{0}_{\text{Riemannian geodesic}} + \underbrace{\frac{Q}{a^2} \nabla_g Q}_{\text{orbital deviation}}$$

When $Q \equiv 0$, orbital geodesics reduce to Riemannian geodesics of (M, g) . The orbital invariant Q introduces a force $F_O = \nabla_g Q^2/2a^2$ that deviates trajectories from free geodesic motion.

2 Jacobi Fields and Stability

Theorem 2 (Orbital Jacobi equation). *Let $\{\gamma_s\}$ be a smooth family of orbital geodesics and $J = \partial_s \gamma_s|_{s=0}$ the associated Jacobi field. Then J satisfies:*

$$\frac{d^2 J}{dt^2} = R_g(\dot{\gamma}, J)\dot{\gamma} + \frac{\langle \nabla_g Q, \dot{\gamma} \rangle_g}{a^2} \nabla_g Q + \frac{Q}{a^2} \text{Hess}_g(Q) \cdot \dot{\gamma}$$

where R_g is the Riemann curvature tensor of (M, g) and $\text{Hess}_g(Q) = \nabla_g^2 Q$ is the Riemannian Hessian of Q .

Corollary 1 (Stability criteria). *Taking the inner product of the Jacobi equation with J , stability is governed by:*

$$\frac{d^2}{dt^2} \frac{1}{2} |J|_g^2 = -K_g(\dot{\gamma}, J) |\dot{\gamma}|_g^2 + \frac{Q}{a^2} \langle \text{Hess}_g(Q) \cdot \dot{\gamma}, J \rangle_g + \frac{\langle \nabla_g Q, \dot{\gamma} \rangle_g}{a^2} \langle \nabla_g Q, J \rangle_g$$

Sufficient condition for focusing (convergence): $K_g > 0$ and $\text{Hess}_g(Q) \leq 0$ (positive sectional curvature, Q concave). Sufficient condition for defocusing (divergence): $K_g < 0$ and $\text{Hess}_g(Q) \geq 0$ (negative sectional curvature, Q convex).

3 Orbital Geodesics on S^2

Definition 2 (Setup on S^2). On S^2 of radius R with the round metric g , take $f(\theta, \phi) = e^{\cos \theta}$, $P = 1$, $b = 0$. Then:

$$Q(\theta, \phi) = \ln f = \cos \theta, \quad \nabla_g Q = -\frac{1}{R} \hat{\theta}$$

The orbital geodesic equation:

$$\frac{D\dot{\gamma}}{dt} = \frac{\cos \theta}{a^2 R} \hat{\theta}$$

Theorem 3 (Three orbital regimes on S^2). For the choice $f = e^{\cos \theta}$ on S^2 :

North pole ($\theta = 0$, $Q = 1 > 0$): The force $\frac{1}{a^2 R} \hat{\theta}$ points toward increasing θ . Geodesics are deflected away from the north pole — it is an orbital repeller of the trajectory, since $\hat{\theta}$ points south.

Equator ($\theta = \pi/2$, $Q = 0$): No orbital force. The equator is a great circle (Riemannian geodesic) and $Q = 0$ there, so $D\dot{\gamma}/dt = 0$. The equator is a free orbital geodesic.

South pole ($\theta = \pi$, $Q = -1 < 0$): The force $-\frac{1}{a^2 R} \hat{\theta}$ points toward decreasing θ , i.e. toward the equator. The south pole is an orbital attractor.

Lemma 1 (Equator as free orbital geodesic). The equator $\{\theta = \pi/2\}$ of S^2 with $f = e^{\cos \theta}$ is simultaneously a Riemannian geodesic of (S^2, g) and a free orbital geodesic ($Q = 0$, zero orbital force). It is the unique great circle with this property for this choice of f .

4 Orbital Fermat Principle and Conformal Metric

Theorem 4 (Orbital Fermat principle). The orbital geodesics on (M, g) are the extremals of the functional:

$$\mathcal{F}[\gamma] = \int_{\gamma} \frac{|Q(\gamma(t))|}{a} ds_g$$

where $ds_g = |\dot{\gamma}|_g dt$ is the Riemannian arc length element. The orbital invariant $|Q|/a$ plays the role of the refractive index in the orbital analogue of Fermat's principle.

Definition 3 (Conformal orbital metric). The functional $\mathcal{F}[\gamma]$ is the arc length for the conformal orbital metric:

$$\tilde{g} = \frac{Q^2}{a^2} g$$

Orbital geodesics are exactly the geodesics of $\tilde{g}$.

Theorem 5 (Curvature of the conformal orbital metric). On a surface (M^2, g) , the Gaussian curvature of $\tilde{g} = (Q^2/a^2)g$ is:

$$\tilde{K} = \frac{a^2}{Q^2} (K_g - \Delta_g \ln |Q|)$$

where K_g is the Gaussian curvature of (M, g) .

Proof. Under a conformal change $\tilde{g} = e^{2\omega} g$ with $e^{2\omega} = Q^2/a^2$, i.e. $\omega = \ln |Q| - \ln a$: $\tilde{K} = e^{-2\omega} (K_g - \Delta_g \omega) = (a^2/Q^2) (K_g - \Delta_g \ln |Q|)$. $\square$

Corollary 2 (Flat orbital metric). The conformal orbital metric $\tilde{g}$ is flat ($\tilde{K} = 0$) when:

$$\Delta_g \ln |Q| = K_g$$

This is a partial differential equation for Q (equivalently for f) in terms of the geometry of (M, g) . On flat (M, g) ($K_g = 0$): $\tilde{g}$ is flat iff $\ln |Q|$ is harmonic, i.e. $|Q|$ is log-harmonic.

5 Conformal Collapse at Zeros of Q

Theorem 6 (Zeros of Q as conformal collapse points). *At a zero x_0 of Q (i.e. $f(x_0) = e^{bP(x_0)}$), the conformal factor Q^2/a^2 vanishes. For a curve γ approaching x_0 , the $\tilde{g}$ -length:*

$$\tilde{\ell}(\gamma) = \int_{\gamma} \frac{|Q(\gamma)|}{a} ds_g \rightarrow 0$$

as $\gamma \rightarrow x_0$ (provided Q vanishes to first order and ds_g is finite). The zeros of Q are at $\tilde{g}$ -distance zero from every neighbouring point: they are conformal collapse points of $\tilde{g}$.

Remark 2 (Orbital interpretation). *In the orbital geometry of $\mathcal{O}$, the zeros of Q are the orbital equilibria — points where $f = e^{bP}$, i.e. where the system is in balance. In the conformal orbital metric, these equilibria collapse: all nearby points are identified with them at zero $\tilde{g}$ -distance. The geometry “contracts” around orbital equilibria.*

This is consistent with Paper IV: near zeros of ζ , the orbital energy wells $E \rightarrow -\infty$, and the system is infinitely attracted to the equilibrium. The conformal collapse is the geometric expression of this attraction.

6 Summary of New Results

Result	Statement	Status
Eikonal on (M, g)	$ \nabla_g \phi _g^2 = Q^2/a^2$	Established
Orbital geodesics covariant	$D\dot{\gamma}/dt = (Q/a^2)\nabla_g Q$	Established
Reduction to Riem. geodesics	$Q = 0 \Rightarrow$ free geodesic	Established
Orbital Jacobi equation	Riemann + Hess(Q) corrections	Established
Focusing criterion	$K_g > 0$, Hess(Q) ≤ 0	Established
Defocusing criterion	$K_g < 0$, Hess(Q) ≥ 0	Established
On S^2 : three regimes	North repeller, equator free, south attractor	Established
Equator as free geodesic	$Q = 0$ on equator, great circle	Established
Orbital Fermat principle	Extremals of $\int Q /a ds_g$	Established
Conformal orbital metric	$\tilde{g} = (Q^2/a^2)g$	Established
Curvature of $\tilde{g}$	$\tilde{K} = (a^2/Q^2)(K_g - \Delta_g \ln Q)$	Established
Flat orbital metric	$\Delta_g \ln Q = K_g$	Established
Conformal collapse	Zeros of Q at $\tilde{g}$ -distance zero	Established
Collapse = attraction	Geometric expression of orbital wells	Established
Orbital geodesics on CY	Eikonal on Ricci-flat 6-manifolds	Open
$\tilde{g}$ and Gauss–Bonnet	$\int_M \tilde{K} d\tilde{V}_g$ via conformal change	Open

Acknowledgements

This paper develops the orbital geodesic structure of Paper XVII on curved manifolds. The conformal orbital metric $\tilde{g} = (Q^2/a^2)g$ emerged from writing the orbital Fermat functional and recognising it as an arc length for a conformally modified metric. The conformal collapse at zeros of Q — the identification of orbital equilibria with $\tilde{g}$ -distance-zero points — was not anticipated: it followed from the definition of $\tilde{g}$ and the vanishing of Q .

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